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A proposal for relativistic transformations in thermodynamics

G Ares de Parga, B López-Carrera and F Angulo-Brown

Dpto. de Física, Escuela Superior de Física y Matemáticas, Instituto Politécnico Nacional, U.P. Adolfo López Mateos, Zacatenco, México D.F.,C.P. 07738, Mexico

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Abstract

Since the beginning of the past century many proposals for relativistic transformations in thermodynamics have been suggested. A general consensus about this matter has not been reached. In this work, we propose a scheme of thermodynamic relativistic transformations inspired by the Planck–Einstein theory, but changing the relativistic transformation of energy. This change permits the form invariance of thermodynamics. Also by means of finite-time thermodynamics we demonstrate the relativistic invariance of thermal efficiency.

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1. Introduction

Since Planck–Einstein [1] (PE) relativistic thermodynamics appeared, little attention has been paid to it. Indeed, as Tolman [2] mentioned: 'The common lack of familiarity with this branch of relativity has doubtless been due to the absence of physical situations where its applications were necessary'. In 1963, Ott [5] presented a new proposal which fundamentally differed from the previous one concerning the temperature transformation. As a consequence of this, an innumerable set of contributions appeared during the 1960s with different transformation laws. At the end of the 1960s, Balescu [14] claimed the validity of the PE proposal by correcting Ott's [5] method and by showing the non-invariant form of the other theories. Nowadays, stationary astrophysical situations revive the subject. Nevertheless, the temperature transformation of special relativity is still being discussed. Landsberg [3] asserted that no consensus has emerged, except that the law has the form

$$T = T(u) = [\gamma(u)]^a\, T(0) = [\gamma(u)]^a\, T_o = \gamma^a T_o \qquad \text{and} \qquad \mathrm{d}Q = \gamma^a\, \mathrm{d}Q_o, \tag{1}$$

where

$$\gamma = \gamma(u) = \left[1 - \frac{u^2}{c^2}\right]^{-\frac{1}{2}}$$

and with T and Q representing the temperature and the heat, respectively. The subscript $_o$ refers to the function measured in the proper frame, and we are just considering motion of the frame in the x axis. Here 'a' (the Balescu parameter) is not known with certainty and $a = -1$ if the Planck–Einstein [1] (PE) proposal is used, $a = 0$ for the Landsberg [4] (L) one and $a = 1$ under the Ott [5] (O) considerations. Recently, considering the directional temperature [7] and integrating the planckian spectrum over the solid angle, Landsberg and Matsas [6] concluded the non-existence of a universal and continuous temperature relativistic transformation since the result is proportional to the excitation rate of the Unruh–DeWitt [8] detector instead of a typical black-body radiation spectrum. Nevertheless, Sieniutycz [9] still remarks that some authors conclude that equilibrium statistical mechanics cannot provide an unambiguous answer to the relativistic transformation formulae of thermodynamic quantities and, therefore, all of the three kinds of transformations are acceptable. However, it is interesting to note that the three Rorhlich's proposals [10], the so-called apparent [AR], true or covariant [TR] and light [LR] transformations, respectively, have to be taken into account within this debate. Although Rorhlich's treatment of this problem is not complete, one of these last proposals, the AR one, would partially coincide with our proposal AA that we will present in this paper.

In order to show the ambiguity claimed by Sieniutycz [9] and to introduce the time dependence in the theory, section 2 will be advocated to show the invariance of the thermal efficiency of a heat engine, for any working regime for any heat-transfer law independently of the temperature transformation we choose. This will be realized with the aid of the so-called 'g' function of finite-time thermodynamics (FTT) [11, 12]. As a consequence of this, the law of transformation for the work will possess an exact differential for any value of the Balescu [14] parameter 'a'. This last result encourages us to analyse the possibility of changing the PE proposal. Section 3 will be dedicated to discuss the existence or not of a temperature transformation. We start by taking into account the work realized by Landsberg and Matsas [6], where by using the directional temperature [7], the excitation rate of the Unruh–DeWitt detector was obtained for the radiation emitted by a black body which is observed from a moving frame with respect to it. By integrating this result over all the frequencies, we obtain the Stefan–Boltzmann law. So we note that there is no impediment to consider a relativistic temperature transformation. We conclude that the black-body radiation is no longer isotropic looked from a moving system as can be expected. A discussion about the validity of the transformation law for the internal energy is open. Section 4 is devoted to deduce the parameter 'a' by using the free expansion of an ideal gas, obtaining $a = -1$. In section 5, the PE-transformation law for the energy is questioned since, if it is true, the invariant form of thermodynamics will no longer be valid. Particularly, starting from the relation between the Helmholtz free energy and the internal energy, it will be shown that the invariant form of thermodynamics will not be conserved. The same situation will occur with the energy density. We will correct this inconvenience by using a method coming from an idea described in section 3 which we will call the renormalization of thermodynamics. In section 6, we explore the validity rank of our AA proposal leading us to constrain the size of the system with respect the time evolution of it for the sake of applicability of the theory. The simultaneity effect will fundamentally constrain the theory. An analysis of a covariant theory will be exposed in section 7. Rohrlich's concepts, about the volume transformation, will be exposed leading us to discuss the enthalpy as a 0 component of a 4-vector linear momentum in thermodynamics and the renormalization process will be used in general. Finally, in the concluding remarks (section 8), we present the final result showing that the transformation law is simple for renormalized quantities.

It must be pointed out that the purpose of this paper is to construct a relativistic transformation law in order to obtain an invariant form of relativistic thermodynamics. We avoid, as Balescu suggests, all kind of tricks as slow accelerations. We also respect Landau's [15] concept that a thermodynamic system must consist of bodies which move in whole translation or rotation, concepts that are not considered in many treatments of this question.

2. Invariance of the efficiency

With the purpose of introducing us to the problem, we will analyse a situation where the ambiguity cited above about the value of 'a' is present. Some thermodynamic functions are relativistic invariants, as the pressure and the entropy [2]. Using equation (1), we will demonstrate that the efficiency is a relativistic invariant for any performance regime of a thermal cycle. Indeed, if we consider the transformation of the work for any 'a', we have

$$\mathrm{d}W = \gamma^{a}\,\mathrm{d}W_o + \varphi\,\mathrm{d}\zeta, \tag{2}$$

where $\varphi\,\mathrm{d}\zeta$ represents a differential form of thermodynamic variables. This differential form will be different for each proposal. Indeed, as we will see at the end of the paper, together with the concept of Lorentz contraction, they will represent the source of our proposal. The efficiency η of any cyclic process will be described by

$$\eta = \frac{W}{Q} = \frac{\gamma^{a} W_o + \int^{\text{The cycle}} \varphi\,\mathrm{d}\zeta}{\gamma^{a} Q_o}, \tag{3}$$

with W being the work performed by the working substance along the cycle and Q the absorbed heat. If and only if $\varphi\,\mathrm{d}\zeta$ is an exact differential we can ensure that the efficiency is a relativistic invariant. For the PE model [1] ('$a = -1$'), the work transforms in a non-invariant form [2]:

$$\mathrm{d}W = \gamma^{-1}\,\mathrm{d}W_o - \gamma\frac{u^2}{c^2}\,\mathrm{d}(E_o + P_o V_o), \tag{4}$$

where W, E, P and V represent the work, the internal energy, the pressure and the volume, respectively. In this case, $\varphi\,\mathrm{d}\zeta$ results to be an exact differential and the integral in equation (3) vanishes and consequently the efficiency results in a relativistic invariant. Nevertheless, we cannot ensure that the invariance is present for any transformation law. On the other hand, although Balescu [14] has developed a statistical mechanics where he claims to prove the consistency of the PE proposal [1] with the invariance of the form for classical thermodynamics (we will see that there is a mistake in this assertion), the other proposals allow us to write all these formalisms in the same invariant form as derived by Planck [1] by using simple gauge transformations. Balescu commented that: 'the degeneracy found here may well be due to the fact that in equilibrium theory one main ingredient of relativity—time—does not appear'. During the last three decades a finite-time thermodynamics (FTT) [12] has been developed. One of the main achievements of FTT has been to formulate heat engine models under more realistic conditions than those of classical equilibrium thermodynamics. We believe it represents an excellent frame to study the relativistic transformations if, as Balescu claimed, the time must be included. In this order of ideas, by using FTT, this section will be advocated to demonstrate that the efficiency is an invariant no matter which transformation law is used, that is, for any value of the Balescu parameter 'a'.

The problem of analysing a thermodynamical system by means of relativistic transformations has always been outlined for equilibrium states where we can assign a well-defined value for each thermodynamical function. However, this approach has some problems; for example, due to the simultaneity concept in relativity, for a time-dependent process different values of the temperature can be considered. We will leave this issue to be analysed in section 6 where we will constrain the applicability of the theory.

First of all it has to be noted that it is straightforward to demonstrate that a Carnot cycle in a proper frame transforms to another Carnot cycle when it is observed from a moving frame. As a consequence of this and by using equation (1) the efficiency turns out to be also an invariant independent of the value of 'a'.

The so-called Curzon and Albhorn cycle (CA) [13] is a typical FTT model evolving at finite rates, and including entropy production. In this order of ideas, it is interesting to study its efficiency. The most simple endoreversible thermodynamic system consists of two reservoirs at temperatures T_1 and T_2 ($T_1 > T_2$) and two thermal resistances with the same thermal conductances connected with a Carnot engine. When we analyse the efficiency at the maximum power regime also turns out to be [13] an invariant.

The last examples encourage us to think that the invariance of the efficiency is independent of the form of the relativistic transformation. De Vos [12] and separately Arias-Hernández and Angulo-Brown [11] have developed the so-called 'g' function [11], which permits one to obtain a general expression for the engine's power output independent of any heat-transfer law. Indeed, if we refer to the so-called g function [11] , we will be able to demonstrate the general scalar behaviour of the efficiency. This g function has been constructed using the first and second laws of thermodynamics and it represents an excellent choice to test the viability of our proposal. $g(\eta)$ is given by [11]

$$g(\eta) = \frac{T_1 T_2 \eta}{T_1 - T_2 - \eta T_1}. \tag{5}$$

For a thermodynamic cycle working between two reservoirs, the power output Π and the universe's entropy production σ for any working regime are related by [11]:

$$\Pi = \frac{Q_1 - Q_2}{\Delta t} = g(\eta)\sigma = \frac{T_1 T_2 \eta}{T_1 - T_2 - \eta T_1}\sigma, \tag{6}$$

with Q_1 and Q_2 being the heats entering and leaving the working fluid, respectively, and where Δt represents the cycle period. From equations (6) and (1), it is clear that Π and σ transform as γ^{a-1} and γ^{-1}, respectively, and therefore g must transform as γ^{a}. From equation (5), we immediately conclude that η is a relativistic invariant, for any working regime of the cycle (such as the maximum power regime, the ecological regime [11] and others). This will imply that the relativistic transformation law for the work will be of the form described by equation (2) with the property that $\varphi\, \mathrm{d}\zeta$ is an exact differential for any value of the parameter 'a'. As we have mentioned in the introduction, in the case of the PE proposal the existence of an exact differential will be the source of much confusion. Although we have proved that the efficiency is invariant for a two-reservoir system, it is clear that for other kinds of thermodynamic cycles the result will persist. Thus, we can conclude that the efficiency is invariant for any transformation law, even if the processes are time dependent. An interesting fact is that FTT predicts that the efficiency is an invariant and as a consequence of this the transformation for the work will always be followed by an exact differential.

As a first comment, we can ensure that any new temperature transformation has to include an exact differential form of thermodynamic quantities which will not realize any work during a cycle. This permits some freedom in the sense that any proposal to which an exact differential is added, will give the same final work of the system and from a thermodynamical point of view will be equivalent. In this sense, the set of possibilities is open since any proposal could be modified just by adding an exact differential. This kind of gauge theory is similar to the one proposed by Balescu [14], but as he claimed the only one which preserves the invariant form of classical thermodynamics is the PE proposal. Nevertheless, in section 5, we will observe that this invariance is not quite exact as Balescu asserts.

Secondly, the existence of a new term in the transformation of the work suggests to us that there is a hidden covariant form included in the theory. Indeed, this has been proposed in many articles leading to different proposals. The interesting work realized by Ott [5] incorporates this idea which gives a different proposal than the PE one [1]. An interesting result can be found in a paper of Rohrlich [10] where an analogy with classical electrodynamics is made. An analysis of this will be realized in section 7.

3. Black-body radiation

In the previous section we pointed out the ambiguity for the choice of the Balescu parameter 'a', as the law of transformation was just a matter of convention. Nevertheless, a question has to be answered: that is, the physical validity of defining a transformation law. Indeed, Landsberg and Matsas [6] claimed the non-existence of such a transformation. The reasoning is as follows: starting by considering a black body at rest in a system K_o at temperature T_o, the particle number density is a Planckian leading to a Stefan–Boltzmann law after integrating over all the solid angle and over all the frequencies. When the black body is observed from a moving frame K, the particle number density is [6, 7]

$$n(w, T_\theta)\,\mathrm{d}w\,\mathrm{d}\Omega = \frac{w^2/c^3}{2\pi^2\big(\big(\exp\frac{\hbar w}{kT_\theta}\big) - 1\big)}\,\mathrm{d}w\,\mathrm{d}\Omega, \tag{7}$$

where T_θ denotes the directional temperature defined as

$$T_\theta = \frac{T_o\sqrt{1-\frac{u^2}{c^2}}}{1-\frac{u}{c}\cos\theta}, \tag{8}$$

with T_o, u, θ, being the temperature in the system K_o, the speed between both frames and the angle between the axis of motion and the direction of observation in the frame K, respectively. The result of integrating the particle number density over all the solid angle and considering the two independent polarizations gives

$$n(w, T_o, u)\,\mathrm{d}w = \frac{wkT_o\sqrt{1-\frac{u^2}{c^2}}}{2\pi^2c^2u\hbar}\ln\frac{1-\exp-\left(\frac{\hbar w\sqrt{1+\frac{u}{c}}}{kT_o\sqrt{1-\frac{u}{c}}}\right)}{1-\exp-\left(\frac{\hbar w\sqrt{1-\frac{u}{c}}}{kT_o\sqrt{1+\frac{u}{c}}}\right)}\,\mathrm{d}w, \tag{9}$$

which is proportional to the excitation rate of the Unruh–DeWitt detector [6, 7]. Landsberg and Matsas [6] assert that since the result is not of a Planckian form, it is impossible to define a temperature transformation law. Moreover, making different averages of the directional temperature, the result is not consistent in order to define a universal temperature. Nevertheless, first of all it has to be noted that if a black body is at rest in a frame, the particle number density and the energy density around a frequency have not to be of a Planckian form in a moving system. Indeed, the non-isotropy of the situation will forbid such a functional form. On the other hand, what has to be conserved is the number of particles and the total energy will indicate the way of transforming the energy from a relativistic mechanical point of view. If such a transformation is adequate, a possible law of temperature may exist. If we integrate equation (9) over all the frequencies and over the volume, the result is

$$N = V\int_0^\infty n(w, T_o, u)\,\mathrm{d}w = \frac{2\zeta(3)}{\pi^2}\left(\frac{T_o}{\hbar c}\right)^3, \tag{10}$$

as expected [16] since the number of particles is the same no matter which frame we choose. This last result is not in contradiction with Landsberg and Matsas [6], since it only describes the conservation of the particles. But if we analyse the energy rate, the result is quite surprising; indeed, if we calculate the energy by integrating the particle number density multiplied by the quantum of energy over all the frequencies and the volume, we obtain

$$E = V \int_0^\infty \hbar w n(w, T_o, u)\, \mathrm{d}w = \frac{4}{c} V \sigma T_o^4 \gamma^2 \left(1 + \frac{u^2}{3c^2}\right). \tag{11}$$

If we compare this result with the PE transformation law of the energy, we will note that it coincides [2] since for the PE proposal E is given by

$$E = \gamma E_o \left(1 + \frac{u^2}{3c^2}\right), \tag{12}$$

where $E_o = \frac{4}{c} V_o \sigma T_o^4$ and noting that $V = \gamma^{-1} V_o$. What we have demonstrated is that the PE transformation law of the energy is consistent with the excitation rate of the Unruh–DeWitt detector. Thus, it will always be possible to define a temperature transformation law which coincides with the PE proposal.

Some hidden aspects in this theory have to be noted. First of all, we note that the energy density is no longer an invariant. Indeed, since the isotropy has been lost, the pressure is different from the third part of the energy density, ε. That is

$$\varepsilon = \frac{E}{V} = \frac{\gamma^2 E_o}{V_o} \left(1 + \frac{u^2}{3c^2}\right) = \gamma^2 \left(1 + \frac{u^2}{3c^2}\right) \varepsilon_o. \tag{13}$$

Since $P = P_o$, this implies that

$$\varepsilon \neq 3P \qquad \text{and} \qquad 3P_o = \varepsilon_o = \frac{4}{c} \sigma T_o^4. \tag{14}$$

By using equation (13), we arrive at

$$\varepsilon = \gamma^2 \left(1 + \frac{u^2}{3c^2}\right) \varepsilon_o = \gamma^2 \left(1 + \frac{u^2}{3c^2}\right) \frac{4}{c} \sigma T_o^4. \tag{15}$$

This would not imply that the temperature transformation law is not well defined, but that the black-body radiation observed from a moving system is deformed in such a way that Stefan–Boltzmann law is not satisfied due the lack of isotropy in this frame. The question now is if the total energy and the energy density obtained in equations (12) and (13) represent good candidates for a thermodynamical theory. Certainly E has been obtained from a relativistic mechanical transformation of the total energy E_o, but, as in the nonrelativistic case [17], the energy of the whole motion has to be eliminated; that is, in order to define a thermal energy in kinetic theory, the averaged velocity of the system must be subtracted from the velocity of each particle. This means the thermal energy is defined by

$$\chi = \tfrac{1}{2} m [\vec{v} - \vec{u}(\vec{r} \cdot t)]^2, \tag{16}$$

where $\vec{u}$ represents the averaged velocity of the system $\vec{u}(\vec{r}, t) = \langle \vec{v} \rangle)$. Returning to our problem, it is evident that we have not considered this aspect in the definition of the total energy and consequently in the energy density . Thus, instead to deal both with the regular energy and its energy density, it will be better to consider a thermodynamical function what we will call the thermal energy ξ and its thermal energy density ς. This question will be analysed in section 6, but as an intuitive deduction let us deal with the black-body situation. First of all, let us accept the PE transformation law for the temperature and the heat to be valid. Secondly, let us propose that the thermal energy and the thermal energy density are defined as follows:

$$\xi = E - W_{0 \to E_o} \qquad \text{and} \qquad \varsigma = \frac{\xi}{V} = \varepsilon - \frac{W_{0 \to E_o}}{V}, \tag{17}$$

where $W_{0\to E_o}$ represents the work which brings the black body in the moving frame from an initial energy 0 to a final energy E_o (this last one measured in the rest frame). The energy $W_{0\to E_o}$ will be called the bulk energy. To be able to calculate such a term it is necessary to understand that during the process the velocity of the system is constant ($\overrightarrow{u} = \overrightarrow{\text{const}}$). Using the momentum of the whole black body which is equal to [2]

$$\overrightarrow{G} = \frac{4}{3}\gamma E_o \frac{\overrightarrow{u}}{c^2}. \tag{18}$$

It is easy to obtain $W_{0\to E_o}$

$$W_{0\to E_o} = \int \overrightarrow{u} \circ \mathrm{d}\overrightarrow{G} = \int_0^{E_o} \frac{4}{3}\gamma \frac{u^2}{c^2}\,\mathrm{d}E_o = \frac{4}{3}\gamma E_o \frac{u^2}{c^2}. \tag{19}$$

So returning to the thermal energy and the thermal energy density, we arrive at

$$\xi = E - \frac{4}{3}\gamma E_o \frac{u^2}{c^2} = \gamma E_o\left(1 + \frac{u^2}{3c^2}\right) - \frac{4}{3}\gamma E_o \frac{u^2}{c^2} = \gamma^{-1} E_o = \gamma^{-1}\xi_o \tag{20}$$

and

$$\varsigma = \frac{\gamma^{-1} E_o}{\gamma^{-1} V_o} = \frac{E_o}{V_o} = \varepsilon_o = \varsigma_o.$$

With this definition of the thermal energy and the thermal energy density, we obtain a simple transformation law. The question now is if this is useful from a thermodynamical point of view. This will be analysed in section 5. It has to be pointed out that in the rest frame the definition of regular and thermal energies coincide and that the specific variable, in this case the thermal energy density ς, is an invariant. The Stefan–Boltzmann law can now be expressed as

$$\varsigma = \varsigma_o = \varepsilon_o = \frac{4}{c}\sigma T_o^4 = \frac{4}{c}\sigma \gamma^4 T^4, \tag{21}$$

and we can now ensure that

$$\varsigma = 3P. \tag{22}$$

It is interesting to note how the structure of thermodynamics is conserved. So with the use of our thermal energy density, we recover a regular form for the expression of the black-body radiation. If we look for the transformation law for the thermal energy, it coincides with the AR proposal for the transformation of the energy. This encourages us to think that our proposal has some sense. In section 5, we will generalize this idea for any system and we will note that the method of subtracting the bulk energy will be related with the form invariance of thermodynamics. This method will be called the renormalization of thermodynamics. Before we generalize this concept, in section 5, we are obliged to demonstrate why the PE proposal for transformation of the temperature is the correct one.

4. Free expansion of ideal gases

As we noted in the introduction, different proposals have appeared after the PE theory. First of all, let us reproduce a table which describes the relativistic transformation laws for different thermodynamical functions. We will partially reproduce a table presented by Balescu [14], where just the five main proposals are kept and we anticipate our proposal.

It is interesting to note that the transformations for the volume (V) are the same in all of the cases except in the TR and LR proposals. This interesting difference will be widely discussed in section 7, since the geometry of classical relativity and thermodynamics would have to be revisited under this idea. The pressure (P) and the entropy (S) are invariant in

Table 1. Relativistic transformation laws for different proposals. N.R. means not reported. The subscript $_o$ describes the variable in the proper system. PE: Planck–Einstein; O: Ott; L: Landsberg; AR apparent Rohrlich; TR: true or covariant Rohrlich; LR: light Rohrlich and AA: Ares de Parga and Angulo-Brown. Note that for the AA proposal the energy is substituted by the thermal energy.

a	V	P	S	T	$\mathrm{d}Q$	E	F
PE	$\gamma^{-1}V_o$	P_o	S_o	$\gamma^{-1}T_o$	$\gamma^{-1}\,\mathrm{d}Q_o$	$\gamma\left(E_o + \frac{u^2 P_o V_o}{c^2}\right)$	$\gamma^{-1}F_o$
O	$\gamma^{-1}V_o$	P_o	S_o	γT_o	$\gamma\,\mathrm{d}Q_o$	γE_o	γF_o
L	$\gamma^{-1}V_o$	P_o	S_o	T_o	$\mathrm{d}Q_o$	E_o	F_o
AR	$\gamma^{-1}V_o$	P_o	S_o	$\gamma^{-1}T_o$	N.R.	$\gamma^{-1}E_o$	N.R.
TR	γV_o	P_o	S_o	γT_o	N.R.	γE_o	N.R.
LR	V_o	P_o	S_o	$\gamma^{-1}T_o$	N.R.	$\gamma^{-1}E_o$	N.R.
				$\xi = E - \gamma(E_o + PV_o)\frac{u^2}{c^2}$			
a	V	P	S	T	$\mathrm{d}Q$	ξ	F
AA	$\gamma^{-1}V_o$	P_o	S_o	$\gamma^{-1}T_o$	$\gamma^{-1}\mathrm{d}Q_o$	$\gamma^{-1}\xi_o = \gamma^{-1}E_o$	$\gamma^{-1}F_o$

all the proposals [14]. But for the internal energy, the expressions fundamentally differ in the value of what we will call the Balescu parameter 'a'. Nevertheless, for the temperature (T), the flux of heat ($\mathrm{d}Q$) and the free Helmholtz energy (F), the laws have a common form; actually, we can always describe them as

$$\Lambda = \gamma^a \Lambda_o, \tag{23}$$

where Λ maybe T, $\mathrm{d}Q$ or F and 'a' is the Balescu parameter [14] which as we noted before for Planck–Einstein [1] is -1, for Ott [5] 1 and for Landsberg [4] 0.

Let us now apply these relations for the free adiabatic expansion of an ideal gas. First of all, we recall that experimentally all gases behave in a universal way when they are sufficiently diluted. The ideal gas is an idealization of this limiting behaviour and with it we can define an absolute scale of temperature. For an ideal gas in its proper frame, we know that

$$P_o V_o = N_o \kappa T_o, \tag{24}$$

where P_o, V_o, N_o, T_o and κ represent, respectively, the pressure, the volume, the number of molecules, the temperature and the Boltzmann constant. If we use equation (1) and if we consider that the volume transforms as

$$V = \gamma^{-1} V_o, \tag{25}$$

which is a consideration accepted for all the proposals described by table 1 except the TR and LR proposals, we obtain

$$PV = P_o \gamma^{-1} V_o = N_o \kappa \gamma^{-1} T_o = N_o \kappa \gamma^{-1-a} T, \tag{26}$$

where $N = N_o$, since the number of molecules is conserved. If we use this last result for calculating the entropy for an adiabatic free expansion with the invariance of the entropy, we arrive at

$$N\kappa \ln \frac{V_{of}}{V_{oi}} = \Delta S_o = \Delta S = N\gamma^{-a-1}\kappa \ln \frac{V_f}{V_i}, \tag{27}$$

where the subscripts $_i$ and $_f$ denote the initial and final states, respectively. Simplifying this expression, we get

$$\frac{V_{of}}{V_{oi}} = \left[\frac{V_f}{V_i}\right]^{\gamma^{-a-1}} = \left[\frac{\gamma V_{of}}{\gamma V_{oi}}\right]^{\gamma^{-a-1}} = \left[\frac{V_{of}}{V_{oi}}\right]^{\gamma^{-a-1}}. \tag{28}$$

It is clear that equation (28) is satisfied if

$$\gamma^{-a-1} = 1 \quad \Longrightarrow \quad a = -1. \tag{29}$$

We can now conclude, independently of the black-body case, that the Balescu parameter $a = -1$ corresponds to the Planck's [1] result. Until now, since the black-body and the free expansion cases are compatible with the Balescu parameter $a = -1$, we can discard the other transformation laws, that is the Ott and Landsberg ones. Nevertheless, it is important to consider the following point: Balescu [14] claimed that 'any of the generalized thermodynamic formalisms can be reduced to Planck's form-invariant formalism by means of a gauge transformation of the temperature and the free energy'. It seems that the main point of the problem consists in choosing the gauge which simplify the particular case we are dealing, as in classical electrodynamics, the freedom of this gauge will be connected with the convenience for dealing with a particular problem. Nevertheless, the deduction of the Balescu parameter has been done by considering the general transformation given by equation (23) and the only consistent choice was for the value $a = -1$. We have to emphasize that only proposals which consider $a = -1$ could lead up to a consistency with thermodynamics when equation (25) has also been considered. Thus, other proposals could be considered provided that a different equation is used for the transformation of the volume as Rohrlich done [10]. Indeed, the TR and LR proposals will give a congruent result in the sense that if instead of using equation (25), we consider $V' = \gamma V_o$ for TR and $V = V_o$ for LR, after repeating all of the above calculation for the free expansion gas, we would obtain $a = 1$ and $a = 0$, respectively. So the point is: which is the good transformation for the volume? We can conclude that this gauge transformation comes from correcting the volume transformation and for this reason the invariance of the form disappears in such cases. We will discuss this point in section 7.

5. Transformation for the Helmholtz free energy

As we noted for the black-body case (section 3), the so-called thermal energy represents a good candidate for describing a thermodynamical system. But it is not just an artificial concept since it comes from subtracting the energy of the motion of the system, which we call the bulk energy of the system. Moreover, the PE proposal must apparently be modified in order to preserve the invariance of the theory, otherwise some thermodynamic functions may be defined with a dependence with respect to the motion of the system and some inconsistencies appear. Indeed, we have pointed out this behaviour with respect to the energy density and the energy for the black-body case, but not for a general situation. The so-called Helmholtz free energy being a thermodynamical potential will permit us to demonstrate that the renormalization method is appropriate for any situation when we are dealing with relativistic thermodynamics. Let us recall the classical relation between the Helmholtz free energy F, the temperature and the entropy [19]:

$$F = E - TS. \tag{30}$$

We require that this last classical relation be preserved in order to obtain an invariant-form relativistic thermodynamics. By using equation (23), and noting that the Helmholtz free energy transforms as [14]

$$F = \gamma^{-1} F_o, \tag{31}$$

we obtain

$$E = \gamma^{-1}(F_o + T_o S) = \gamma^{-1} E_o. \tag{32}$$

But if we review table 1, we note that for the PE proposal, the expression for the transformed energy differs from the result of equation (32) and coincides with the other cases $a = 0$ and $a = 1$, that is, for Landsberg and Ott's proposals which have already been discarded. For TR and LR proposals there is no contradiction provided that the corresponding transformation of the volume has been considered, that is $V = \gamma V_o$ and $V = V_o$, respectively. This is one of the reasons why there exist such proposals. As in the black-body case, if we persist in keeping the PE proposal, we will be obliged to modify the expression of the Helmholtz free energy. So, we can conclude that the concept of regular energy has to be abandoned and substituted by the thermal energy ξ and the thermal energy density ς in order to preserve the invariant form of thermodynamics. Let us define in a general form the thermal energy ξ,

$$\xi = E - W_{0\to E_o}, \tag{33}$$

where $W_{0\to E_o}$ represents the energy needed to bring the system from an energy 0 to E_o but leaving u as a constant:

$$W_{0\to E_o} = \int_0^{E_o} \overrightarrow{u} \circ \mathrm{d}\overrightarrow{G} = \int_0^{E_o} \overrightarrow{u} \circ \mathrm{d}\left[\gamma(E_o + PV_o)\frac{\overrightarrow{u}}{c^2}\right], \tag{34}$$

where we have taken the classical result for the momentum of the system [2]:

$$\overrightarrow{G} = \gamma(E_o + PV_o)\frac{\overrightarrow{u}}{c^2}, \tag{35}$$

and considering that $\overrightarrow{u}$ is a constant, we arrive at

$$\xi = E - \gamma\frac{u^2}{c^2}(E_o + PV_o). \tag{36}$$

After a simple algebra considering the PE transformation for the energy, we obtain

$$\xi = \gamma\left(E_o + \frac{u^2}{c^2}PV_o\right) - \gamma\frac{u^2}{c^2}(E_o + PV_o) = \gamma^{-1}E_o = \gamma^{-1}\xi_o. \tag{37}$$

We substitute E by ξ in equation (30), we arrive at

$$F = \xi - TS = \gamma^{-1}E_o - \gamma^{-1}T_oS = \gamma^{-1}(E_o + T_oS) = \gamma^{-1}F_o, \tag{38}$$

and equation (31) is respected. For the thermal energy density ς, respecting the philosophy of equation (17), the result is

$$\varsigma = \frac{\xi}{V} = \frac{\gamma^{-1}\xi_o}{\gamma^{-1}V_o} = \frac{\xi_o}{V_o} = \varsigma_o = \varepsilon_o. \tag{39}$$

In fact, if we review the Balescu approach [14], one can note that our scheme of renormalizing or of defining the thermal variables is equivalent to not considering the motion of the system as a changing variable. Planck's proposal is not congruent with the invariant form of thermodynamics as was detected in equation (32). So our AA proposal (see table 1) modifies the PE one just in the introduction of the thermal energy instead of the regular energy, leading to an invariant-form relativistic thermodynamics. It has to be noted that Rohrlich [10] has developed a similar proposal (AR) but with the inconvenience of being incomplete since it does not express the transformation laws for the heat and the free Helmholtz energy.

Let us now define other thermodynamical functions as the 'thermal' work. Indeed if we want the first law of thermodynamics to be valid, we need to define a 'thermal' work; that is,

$$\mathrm{d}E = \mathrm{d}Q - \mathrm{d}W \Rightarrow \mathrm{d}E - \overrightarrow{u} \circ \mathrm{d}\overrightarrow{G} = \mathrm{d}Q - \mathrm{d}W - \overrightarrow{u} \circ \mathrm{d}\overrightarrow{G}, \tag{40}$$

and the 'thermal' work is

$$\mathrm{d}\Omega = \mathrm{d}W + \overrightarrow{u} \circ \mathrm{d}\overrightarrow{G} \quad \Rightarrow \quad \mathrm{d}\xi = \mathrm{d}Q - \mathrm{d}\Omega. \tag{41}$$

It is very important to note two aspects; first of all that the 'thermal' work $d\Omega$ transforms as

$$d\Omega = \gamma^{-1}\, d\Omega_o = \gamma^{-1}\, dW_o, \tag{42}$$

and secondly that the additional term used to define the thermal work, equation (41), coincides with the exact differential described in equation (4),

$$\vec{u} \circ d\vec{G} = d\left[\gamma(E_o + PV_o)\frac{u^2}{c^2}\right]. \tag{43}$$

This is because u has to be considered as a constant. In the same order of ideas, it is interesting to note that, as we comment in section 2, this exact differential does not produce any work over a cycle. This was the main point in using FTT since it permitted us to analyse this exact differential which leads us to the renormalization.

6. Constraint of the theory

First of all, it has to be pointed out that our problem has to be revisited with respect to the simultaneity concept in relativity. Indeed if we are dealing with a thermodynamical system where the process depends on time, we measure the volume V_o at a time t_o. In the moving frame the measuring of the volume V at a time t corresponds to different times t_o' and t_o in the proper frame and vice versa. For a time-dependent process, the temperature changes during the process in the different frames. If we measure the temperature T_o at an instant t_o , at two different points x_{o1} and x_{o2} in the rest frame, the corresponding events $(x_{o1}, t_o, T_o) \to (x_1, t_1, T_1)$ and $(x_{o2}, t_o, T_o) \to (x_2, t_2, T_2)$, in the moving frame, would be such that $t_1 \neq t_2$ and $T_1 \neq T_2$. Which will be the good temperature in the moving frame if the temperature in the rest frame is, for any x_o, the same T_o? That is, $T = \gamma^a T_o$ corresponds to the temperature at the moving frame at time t_1 or t_2 ($T_1 = T$ or $T_2 = T$). This is what we call the simultaneity effect. Obviously, there is an indetermination which can be solved by means of considering that the thermodynamical system is small enough to be able to approximate $t_1 = t_2$.

We have to point out that until now all the processes in the different papers on this matter have considered only equilibrium states which, as has been demonstrated by Pia and Balescu [20], are Lorentz invariants; that is, the simultaneity difficulties do not appear. So the AR, TR, LR and AA proposals are limited to small systems where the difference due to the non-single definition of the thermodynamical functions can be negligible. This does not represent an important problem since in reality what we are pursuing is the basic statements for a relativistic hydrodynamics at a mesoscopic level.

Another point that has to be analysed is the way we can understand the relation of the temperatures between different inertial systems. The law of transformation is not transitive. Indeed let us consider two systems, K and K', with respect the rest frame K_o of the thermodynamical system, with velocities with respect K_o, u and v, respectively. The law of transformation is therefore $T = \gamma_u^{-1} T_o$ and $T' = \gamma_v^{-1} T_o$, which does not imply that $T' = \gamma_v^{-1}\gamma_u T$. The good transformation of temperatures between frames K' and K is $T' = \gamma_w^{-1} T$, with w being the velocity between the frames K and K'.

7. Rohrlich's proposals

As we noted before, Rorhlich has proposed that the fundamental point of the relativistic thermodynamical theory is to understand which transformation for the volume has to be used. He proposed [10]

$$V = \gamma^s V_o, \tag{44}$$

where s will be called the Rohrlich parameter and it will be equal to -1 for the PE, AR and AA proposals, 1 for the TR proposal and 0 for the LR proposal. If we reproduce all the calculations for deducing the Balescu parameter using $s = 1$ or $s = 0$, we obtain $a = 1$ or $a = 0$, respectively. The discussion is now which is the physical value of s. Since there is a natural transformation of the volume in a covariant form for Rohrlich, the good value of s is 1 (TR proposal). Furthermore, since the physical image of a system can be connected with a photograph, the LR proposal, $s = 0$, is acceptable and it just consists of considering the thermodynamical variables measured in the rest frame. The question of which is the transformation for the volume that has to be accepted in relativistic thermodynamics can just be answered within the concepts of measurement and invariance. Indeed, if we consider the scalar behaviour of the product of the volume and the time, we have

$$\mathrm{d}V\,\mathrm{d}t = \mathrm{d}^4x, \tag{45}$$

which is a relativistic invariant. Since the proper time has the particular meaning of being the time where thermodynamic functions are measured in classical thermodynamics at a rest frame, for this reason it is our point of departure. We know that

$$\mathrm{d}t = \gamma\,\mathrm{d}\tau, \tag{46}$$

where τ is the proper time.

Considering equation (45), this will obligate us to recognize the appropriate transformation for relativistic thermodynamics as

$$\mathrm{d}V = \gamma^{-1}\,\mathrm{d}V_o, \tag{47}$$

leading us to consider that

$$V = \gamma^{-1}V_o, \tag{48}$$

when the dimension of a thermodynamic system permits us to neglect the simultaneity effect. Consequently, with this constraint, we are able to deal with a well-defined relativistic thermodynamics. We can conclude that renormalizing the PE proposal is the correct way of describing relativistic thermodynamics. It always has to be considered that this theory will just have a meaning if the system is sufficiently small compared with the speed of the moving frame in order to neglect the simultaneity effect.

In order to go deeper into the invariance of the form in the AA proposal, it is interesting to analyse the work realized by Rohrlich [10] with respect the PE proposal. He arrives to a covariant form where the enthalpy of the system is used to define a relativistic vector which explains the reason of the appearance of new terms in the energy among other results. Indeed, as we noted in section 2, an exact differential in the transformation of the work will explain an extra term in the transformation of the energy. Moreover, Staruszkiewicz [22] has shown the covariant characteristic of the enthalpy in the sense that it defines a 4-vector as follows:

$$H^\mu = \left(H, \gamma H_o\frac{\overrightarrow{u}}{c^2}\right) = \left(E + PV, \gamma\,(E_o + PV_o)\,\frac{\overrightarrow{u}}{c^2}\right), \tag{49}$$

where $H = E + PV$ represents the enthalpy. Using the transformation for the regular energy E, it is straightforward to show that

$$H^\mu = H_o u^\mu, \tag{50}$$

where H_o represents the enthalpy in the rest frame and u^μ is described by the 4-vector relativistic velocity. Consequently, H^μ is a 4-vector. We note that the enthalpy transforms as

$$H = \gamma H_o. \tag{51}$$

Staruszkiewicz [22] has shown that

$$\mathrm{d}H = \mathrm{d}(E + PV) = \vec{u} \circ \mathrm{d}\vec{G} + \gamma^{-1}T_o\,\mathrm{d}S + V\,\mathrm{d}P, \tag{52}$$

which is equal to

$$\mathrm{d}H = \vec{u} \circ \mathrm{d}\vec{G} + T\,\mathrm{d}S + V\mathrm{d}P. \tag{53}$$

From this, we can now renormalize the enthalpy by defining the thermal enthalpy Ψ as follows:

$$\mathrm{d}\Psi = \mathrm{d}H - \vec{u} \circ \mathrm{d}\vec{G} = T\,\mathrm{d}S + V\,\mathrm{d}P. \tag{54}$$

We finally obtain

$$\Psi = \xi + PV = \gamma^{-1}(\xi_o + PV_o) = \gamma^{-1}(E_o + PV_o) = \gamma^{-1}\Psi_o = \gamma^{-1}H_{o.} \tag{55}$$

An important aspect of renormalizing thermodynamics is that the group of transformation is reduced to a simple one: all the extensive thermodynamic functions transform as

$$\Gamma' = \gamma^{-1}\Gamma_o \tag{56}$$

which is due to the invariant characteristic of

$$\Gamma'\,\mathrm{d}t' = \gamma^{-1}\Gamma_o\gamma\,\mathrm{d}\tau = \Gamma_o\,\mathrm{d}\tau. \tag{57}$$

For the specific variables, we also obtain invariants; that is

$$\phi = \frac{\Gamma'}{V'} = \frac{\gamma^{-1}\Gamma_o}{\gamma^{-1}V_o} = \frac{\Gamma_o}{V_o} = \phi_o, \tag{58}$$

as happened with the efficiency and the thermal energy density.

8. Concluding remarks

Finally, we want to emphasize the following three main points.

(A) It is a well-known result that the virial theorem for relativistic particles has the form [21]:

$$E = \sum_a m_a c^2\sqrt{1 - \frac{v_a^2}{c^2}} \quad \Rightarrow \quad T = \gamma^{-1}T_o. \tag{59}$$

Since temperature is always related with an average of the energy of the particles, it seems that it is very intuitive and natural that the transformation law for the temperature has the form described in equation (59). In fact, this can be the basis for a statistical treatment of the relativistic ideal gas as a complementary deduction of what has been done by Jüttner [23] and Balescu [14] with the additional introduction of the thermal energy described by our process of renormalization.

(B) It is necessary to emphasize that the PE proposal of the transformation of the energy is the source of other theories. Indeed, the O proposal respects a covariant model for the transformation of the energy (see table 1). Nevertheless, it possesses the inconvenience of not preserving the invariance of the form of thermodynamics. The reason is that it does not consider equation (44) with $s = 1$, as Rohrlich has done. For the LR proposal, the non-invariance came from considering equation (25) instead of equation (44) with $s = 0$.

In summary, if we want a relativistic thermodynamic theory with the fundamental property of possessing invariant form, the Balescu parameter depends on the choice for the transformation of the volume. That is, the correct option for $s = 0$ is the LR proposal; for $s = 1$, the TR proposal and for $s = -1$, the AA proposal. Since classical thermodynamics has been defined in the rest frame and naturally the measurement of the volume is realized by using the concept of simultaneity in this frame, we conclude that the AA proposal with the constraint about the size of the system represents the best option.

(C) A natural renormalization has been done in order to preserve the invariance of the form of thermodynamics. This renormalization is not only in the form but is needed for the consistency of the theory since it is the only way of avoiding the problem presented in dealing both with the energy and the Helmholtz free energy (equation (32)). The covariant form is changed since instead of transforming quantities coming from a regular theory of relativity (which are well defined but are inoperative from a thermodynamical point of view), the AA proposal just deals with the invariance of the form exposed in equation (57).

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References

[1] Planck M 1908 *Ann. Phys. (Lpz)* **26** 1
Einstein A 1907 *Jahrb. f. Rad. u. Elekt.* **4** 411
[2] Tolman R C 1934 *Relativity Thermodynamics and Cosmology* (Oxford: Oxford University Press) pp 2, 155–60
[3] Landsberg P T 1980 *Phys. Rev. Lett.* **45** 149
[4] Landsberg P T 1966 *Nature* **212** 571
Landsberg P T 1967 *Nature* **214** 903
Landsberg P T 1966 *Proc. Phys. Soc.* **89** 1007
[5] Ott H 1963 *Z. Phys.* **175** 70
[6] Landsberg P T and Matsas G E A 1996 *Phys Lett* A **223** 401
Landsberg P T and Matsas G E A 2002 *Physica* A **340** 92
[7] Bracewell R N and Conklin E K 1968 *Nature* **219** 1343
Peebles P J E and Wilkinson D T 1968 *Phys Rev.* **174** 2168
Henry G R, Feduniak R B, Silver J E and Peterson M A 1968 *Phys. Rev.* **176** 1451
[8] Unruh W C 1976 *Phys. Rev.* D **14** 870
DeWitt B S 1979 *General Relativity* ed S W Hawking and W Israel (Cambridge: Cambridge University Press)
[9] Sieniutycz S 1998 *Phys. Rev.* E **58** 7027
[10] Rohrlich F 1966 *Nuovo Cimento* B **45** 6200
[11] Arias-Hernández L A and Angulo-Brown F 1997 *J. Appl. Phys.* **81** 2973
[12] De Vos A 1992 *Endoreversible Thermodynamics of Solar Energy Conversion* (New York: Oxford University Press) chapter 3
[13] Curzon F L and Albhorn B 1972 *Am. J. Phys.* **22** 222
[14] Balescu R 1968 *Physica* **40** 309
[15] Landau L D and Lifshiz E M 1958 *Stat. Phys.* (London: Pergamon) chapters 10, 26 and 27
[16] Landau L D and Lifshiz E M 1958 *Statistical Physics* (London: Pergamon) chapter 60
[17] Huang K 1963 *Statistical Mechanics* (New York: Wiley) chapter 5
[18] Rubin M H 1979 *Phys. Rev.* A **19** 1272
[19] Landau L D and Lifshiz E M 1958 *Statistical Physics* (London: Pergamon) chapter 15
[20] Piña E and Balescu R 1968 *Acta Phys. Austr.* **28** 309
[21] Landau L D and Lifshitz E M 1971 *Classical Theory of Fields* (Oxford: Pergamon and Reading, MA: Addison-Wesley) chapter 34
[22] Staruszkiewicz A 1966 *Nuovo Cimento* A **45** 684
[23] Jüttner F 1911 *Ann. Phys.* **34** 856